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Direct approximation theorems for Dirichlet series in the norm of uniform convergence

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Abstract

We consider functions $f \in AC(\overline{D})$ on a convex polygon $D \subset \mathbb{C}$ and their regularity in terms of Tamrazov's moduli of smoothness. Using the relation between Fourier and Leont'ev coefficients given in (CMFT 1(1) (2001) 193) we prove direct approximation theorems of Jackson type for the Dirichlet expansion

$$f(z) \sim \sum_{\lambda \in \Lambda} \kappa_f(\lambda) \frac{e^{\lambda z}}{L'(\lambda)},$$

where $L(z) = \sum_{k=1}^{N} d_k e^{a_k z}$ is a quasipolynomial with respect to the vertices $a_1, \ldots, a_N$ of D and Λ its set of zeros. We show by an example that our results improve Mel'nik's estimates in (Ukrainian Math. J. 40(4) (1988) 382) on the rate of convergence.

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1. Introduction

Let D be an open convex polygon with vertices at the points $a_1, \ldots, a_N$, $N \geqslant 3$, $\overline{D}$ its closure and $\partial D = \overline{D} \setminus D$ the boundary of D. We assume $0 \in D$.

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By $AC(\overline{D})$ we denote the space of all functions f holomorphic in D and continuous on $\overline{D}$ with finite norm of uniform convergence $\|f\|_{AC(\overline{D})} = \max_{z\in\overline{D}} |f(z)| < \infty$. The class $AC^q(\overline{D})$ contains all functions f holomorphic in D with $f^{(q)} \in AC(\overline{D})$.

Consider the quasipolynomial $L(z) = \sum_{k=1}^N d_k e^{a_k z}$, where $d_k \in \mathbb{C} \setminus \{0\}$ and a_k as above, $k = 1, \dots, N$. By Λ we denote the set of zeros λ_m, $m \in \mathbb{N}$, of the quasipolynomial L.

We expand functions $f \in AC(\overline{D})$ with respect to the family $\mathcal{E}(\Lambda) := \{e^{\lambda_m z}\}_{m\in\mathbb{N}}$ into a series of complex exponentials, the so-called Dirichlet series

$$f(z) \sim \sum_{m\in\mathbb{N}} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)}, \tag{1}$$

where

$$\kappa_f(\lambda_m) = \sum_{k=1}^N d_k e^{a_k \lambda_m} \int_{a_j}^{a_k} f(\eta) e^{-\lambda_m \eta}\, d\eta \tag{2}$$

are the Leont'ev coefficients. The indexing in series (1) is chosen such that $|\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$, in (2) we fix $j \in \{1, \dots, N\}$ arbitrarily. Many important results on these series are due to Leont'ev [5].

Dzjadyk showed in [3] (with $d_k = 1$ for all $k = 1, \dots, N$, but this is inessential) that series (1) converges absolutely for all $z \in \overline{D}$ and uniformly to f for every function $f \in AC(\overline{D})$ which satisfies

$$\sum_{j=1}^N f(a_j) = 0 \quad \text{and} \quad \int_0^c \frac{\omega(t)}{t}\, dt < \infty, \quad c = \text{const} > 0.$$

Here $\omega(t) = \omega_{1,\overline{D}}(f,t)_\infty = \sup_{z,w\in\overline{D},|z-w|<t} |f(z) - f(w)|$ denotes the first modulus of continuity of f on $\overline{D}$.

In this paper, we consider functions $f \in AC(\overline{D})$ with certain regularity conditions and the rate of approximation of their Dirichlet expansion. Results for first moduli were proved by Mel'nik [9]. We extend his results to moduli of arbitrary order using Tamrazov's moduli of smoothness and the relation between Leont'ev and Fourier coefficients proved in [4].

The following section gives a closer look at the zeros of the quasipolynomial L. The next section introduces the notion of Tamrazov's moduli. In Section 4 we give Mel'nik's result on the order of approximation with Dirichlet series for first moduli of continuity and extend his theorem to moduli of smoothness of arbitrary order. The respective proofs are presented in Section 5. The last section gives an example on the scope of our result.

2. The set of zeros of the quasipolynomial L

First let us have a closer look at the quasipolynomial

$$L(z) = \sum_{k=1}^N d_k e^{a_k z},$$

where $d_k \in \mathbb{C} \setminus \{0\}$ and a_k as above, $k = 1, \ldots, N$. For the set of zeros Λ of the quasipolynomial L the following results are well known [5 Chapter 1, Section 2, 6]:

1. The zeros $\lambda_n^{(j)}$ of L with $|\lambda_n^{(j)}| > C$ for sufficient large C have the form $\lambda_n^{(j)} = \widetilde{\lambda}_n^{(j)} + \delta_n^{(j)}$, where $\widetilde{\lambda}_n^{(j)} = \frac{2\pi n i}{a_{j+1}-a_j} + q_j e^{i\beta_j}$ and $|\delta_n^{(j)}| \leqslant e^{-an}$. Here $0 < a = \text{const.}$, $j = 1, \ldots, N$, $n > n_0$ and $a_{N+1} := a_1$. The parameters β_j and q_j are given by $e^{q_j (a_{j+1}-a_j) e^{i\beta_j}} = -\frac{d_j}{d_{j+1}}$, where $d_{N+1} := d_1$. Hence these zeros are simple. The set of zeros Λ can be represented in the form
$$\Lambda = \{\lambda_n\}_{n=1,\ldots,n_0} \cup \left(\bigcup_{j=1}^{N} \{\lambda_n^{(j)}\}_{n=n(j),n(j)+1,\ldots} \right).$$
2. There are positive constants A_1 and c_1 such that for all $n \geqslant n(j)$ and all $\xi \in [a_j, a_k]$ we have $|e^{-\lambda_n^{(j)}(\xi - a_k)} - e^{-\widetilde{\lambda}_n^{(j)}(\xi - a_k)}| \leqslant A_1 \cdot e^{-c_1 n}$. Here $[a_j, a_k]$ denotes the straight-line closed interval between the vertices a_j and a_k in the complex plane.
3. There is a constant $c_2 > 0$ such that for all $k \in \mathbb{N}_0$ there exists a positive constant $A(k)$ with
$$\left| \frac{(\lambda_n^{(j)})^k e^{\lambda_n^{(j)} z}}{L'(\lambda_n^{(j)})} - (-1)^n B_j (\widetilde{\lambda}_n^{(j)})^k e^{\widetilde{\lambda}_n^{(j)} \left(z - \frac{a_{j+1}+a_j}{2}\right)} \right| \leqslant A(k) e^{-c_2 n}$$
for all $n > n_0$. Here all $B_j \neq 0$ are constants, $j = 1, \ldots, N$. This inequality is true for all $z \in \overline{D}$.

For simplicity reasons we assume that all zeros of L are simple. We shall use these properties of Λ to estimate the exponentials in partial series.

3. Tamrazov's moduli of smoothness

To get a sophisticated view on the regularity of functions in $AC(\overline{D})$ Tamrazov introduced in [13] appropriate moduli of smoothness. Let $\xi \in \overline{D}$, $r \in \mathbb{N}$, $\delta > 0$ and $A > 0$. Let $U(\xi, \delta) := \{z \in \mathbb{C} : |z - \xi| \leqslant \delta\}$ be the closed δ-ball with center ξ. We denote by $T(\overline{D}, \xi, r, \delta, A)$ the set of all vectors $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$ with

(i) $z_i \in \overline{D} \cap U(\xi, \delta)$ for all $i = 1, \ldots, r$, and
(ii) $|z_i - z_j| \geqslant A\delta$ for all $i \neq j$, $i, j = 1, \ldots, r$.

If there is no vector satisfying these conditions we set $T(\overline{D}, \xi, r, \delta, A) := \emptyset$. Nevertheless for $A = 2^{-r}$ there exists $\delta > 0$ with $T(\overline{D}, \xi, r, \delta, A) \neq \emptyset$. Let $T_1 = T(\overline{D}, \xi, r+1, \delta, 2^{-r})$. Let $L(z, f, z_1, \ldots, z_r)$ be the polynomial in z of degree at most $r - 1$ which interpolates f at the points $z_1, \ldots, z_r$. The rth modulus of f is defined by
$$\begin{aligned}\omega_r(f, t) &= \omega_{r,\overline{D}}(f, t)_\infty \\ &:= \sup_{0<\delta\leqslant t} \sup_{\xi\in\overline{D}} \sup_{\substack{\mathbf{z}\in T_1 \\ \mathbf{z}=(z_0,\ldots,z_r)}} |f(z_0) - L(z_0, f, z_1, \ldots, z_r)|. \end{aligned} \tag{3}$$

Here the supremum over the empty set is defined as zero. Modulus (3) is equivalent to the best-approximation with algebraic polynomials [16,10]

$$\widetilde{\omega}_r(f,t)=\widetilde{\omega}_{r,\overline{D}}(f,t)_\infty:=\sup_{0<\delta\leqslant t}\ \sup_{\xi\in\overline{D}}\ \inf_{P_{r-1}\in\Pi_{r-1}}\|f-P_{r-1}\|_{AC(\overline{D}\cap U(\xi,\delta))},\tag{4}$$

where Π_{r-1} denotes the vector space of all algebraic polynomials of degree at most $r-1$.

For our purposes an easy estimation of moduli (3) and (4) is needed. Tamrazov defined normal majorants φ with exponent γ: These are bounded non-decreasing functions $\varphi:\,]0,\infty[\,\to\,]0,\infty[$ such that for fixed $\sigma\geqslant 1$ and an exponent $\gamma\geqslant 0$ the following normality condition holds:

$$\varphi(t\delta)\leqslant\sigma t^\gamma\varphi(\delta)$$

for all $\delta>0$, $t>1$ [12, Section 1].

Both moduli defined above are normal [14,15, Theorem 1], i.e.,

$$\omega_{r,\overline{D}}(f,t\delta)_\infty\leqslant C\cdot t^r\cdot\omega_{r,\overline{D}}(f,\delta)_\infty\quad\text{and}\quad\widetilde{\omega}_{r,\overline{D}}(f,t\delta)_\infty\leqslant\widetilde{C}\cdot t^r\cdot\widetilde{\omega}_{r,\overline{D}}(f,\delta)_\infty,$$

where $C,\widetilde{C}>0$ depend on r and the polygon D only. Thus normality is preserved while estimating with normal majorants.

With these moduli and majorants we define classes of regularity: By $AH_r^\varphi(\overline{D})$ we denote the class of all functions $f\in AC(\overline{D})$ with $\omega_{r,\overline{D}}(f,t)\leqslant\text{const.}\cdot\varphi(t)$ and by $AW^qH_r^\varphi(\overline{D})$, $q\in\mathbb{N}$, the class of functions f regular on D, such that $f^{(q)}\in AH_r^\varphi(\overline{D})$. We set $AW^0H_r^\varphi(\overline{D})\equiv AH_r^\varphi(\overline{D})$. For intervals I we just write $H_r^\varphi(I)$ resp. $W^qH_r^\varphi(I)$.

We shall use these moduli to state our results in Section 4. Their normality property is essential in Theorem 3 [4].

4. Direct approximation theorems

Mel'nik established in [9] a direct theorem on the approximation of functions regular in D and continuous in $\overline{D}$ by partial series of (1) analogous to the well-known approximation theorems of periodic functions by trigonometric series:

Theorem 1 (*Mel'nik [9, Thm. 1]*). *Let $f\in AW^qH_1^\omega(\overline{D})$, $q\in\mathbb{N}_0$, and a modulus of continuity $\omega=\omega_{1,\overline{D}}$ satisfy the Zygmund condition*

$$\int_0^h\frac{\omega(f^{(q)},t)}{t}\,dt+h\cdot\int_h^{2\pi}\frac{\omega(f^{(q)},t)}{t^2}\,dt\leqslant c\cdot\omega(f^{(q)},h)$$

for all $0<h<2\pi$ and some positive constant c. Let

$$\sum_{k=1}^N d_kf^{(s)}(a_k)=0\quad\textit{for all }0\leqslant s\leqslant q.$$

Let $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$ be a multi-index. Consider the corresponding quasipolynomial of Jackson's type

$$\mathcal{P}_{q,n}(f)(z) := \sum_{m=1}^{n_0} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)} + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,m}^{q+1}) \kappa_f(\lambda_m^{(j)}) \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}.$$

The coefficients $x_m = x_{n_j,m}$ are determined by the Jackson kernel through the relations

$$x_m = 1 - J_m$$

and

$$\frac{3}{2M_j(2M_j^2+1)} \left(\frac{\sin(M_j t/2)}{\sin(t/2)} \right)^4 = \frac{J_0}{2} + \sum_{m=1}^{n_j} J_m \cos(mt),$$

where $M_j = \lfloor \frac{n_j}{2} \rfloor + 1$.

Then

$$\| f - \mathcal{P}_{q,n}(f) \|_{AC(\overline{D})} \leqslant \text{const.} \sum_{k=1}^{N} \frac{1}{n_k^q} \cdot \omega_{1,\overline{D}} \left(f, \frac{1}{n_k} \right)_\infty .$$

For the proof see [9].

Our new result is the extension of Mel'nik's Theorem 1 to arbitrary moduli of smoothness. Let $1 \leqslant j \leqslant N$ be fixed and $r \in \mathbb{N}$. Let $f \in AC(\overline{D})$ have $r-1$ existing derivatives at the vertices a_k, $k = 1, \ldots, N$, of the polygon. Consider for $k \neq j+1$ the polynomial $P_{j,k}$ of degree at most r that interpolates f at the vertices a_j und a_k and $f', \ldots, f^{(r-1)}$ at the vertex a_k. For $k = j+1$ let the polynomial $P_{j,j+1}$ interpolate $f, f', \ldots, f^{(r-1)}$ at both vertices a_j and a_{j+1}. We define

$$\delta_{r,j}(f,h) := \sum_{\substack{k=1 \\ k \neq j}}^{N} \left\{ \int_0^h \frac{\left| f\left(a_k + \frac{a_j - a_k}{2\pi} u\right) - P_{j,k}\left(a_k + \frac{a_j - a_k}{2\pi} u\right) \right|}{u} \, du \right.$$
$$\left. + h^r \cdot \int_h^{2\pi} \frac{\left| f\left(a_k + \frac{a_j - a_k}{2\pi} u\right) - P_{j,k}\left(a_k + \frac{a_j - a_k}{2\pi} u\right) \right|}{u^{r+1}} \, du \right\}$$

and

$$\delta_r(f,h) \quad := \quad \max_{1 \leqslant j \leqslant N} \delta_{r,j}(f,h).$$

For the approximation of $f \in AC(\overline{D})$ we use partial Dirichlet series weighted with the generalized Jackson kernel

$$K_{n,r}(t) := \lambda_{n,r} \left(\frac{\sin Mt/2}{t/2} \right)^{2r} = \frac{J_{n,r,0}}{2} + \sum_{k=1}^{n} J_{n,r,k} \cos kt,$$

where $n \in \mathbb{N}$, $r \geqslant 2$, $M := \lfloor \frac{n}{r} \rfloor + 1$, and $\lambda_{n,r}$ is chosen such that

$$\frac{1}{2\pi}\int_0^{2\pi} K_{n,r}(t)\, dt = 1.$$

$K_{n,r}$ is an even non-negative trigonometric polynomial of degree at most n. We denote by $\mathcal{P}_{q,n,r}(f)$ the quasipolynomial of Jackson's type

$$\begin{aligned}\mathcal{P}_{q,n,r}(f)(z) := &\sum_{m=1}^{n_0} \kappa_f(\lambda_m) \cdot \frac{e^{\lambda_m z}}{L'(\lambda_m)} \\ &+ \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}^{q+1}) \kappa_f(\lambda_m^{(j)}) \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})},\end{aligned}$$

with

$$x_{n_j,r,m} = \sum_{p=0}^{r} (-1)^p \binom{r}{p} J_{n_j,r,mp}.$$

Theorem 2. *Let $f \in AH_r^{\omega_r}(\overline{D})$, $r \geqslant 2$, and ω_r be a normal majorant with exponent r satisfying the Stechkin condition*

$$\int_0^h \frac{\omega_r(t)}{t}\, dt + h^r \cdot \int_h^{2\pi} \frac{\omega_r(t)}{t^{r+1}}\, dt \leqslant c \cdot \omega_r(h) \tag{5}$$

for all $0 < h < \frac{2\pi}{r}$ and a positive constant c. Let $f^{(r-1)}$ be continuous in a neighborhood of the vertices a_k, $k = 1, \ldots, N$, and

$$\sum_{k=1}^{N} d_k\, f^{(s)}(a_k) = 0, \quad 0 \leqslant s \leqslant r-1.$$

Let $n = (n_1, \ldots, n_N) \in \mathbb{N}^N$ be multi-index.

Then for the quasipolynomial $\mathcal{P}_{0,n,r}(f)$ weighted with the generalized Jackson kernel and for some normal majorant Ω_r with exponent r

$$\| f - \mathcal{P}_{0,n,r}(f) \|_{AC(\overline{D})} \leqslant \text{const.} \cdot \sum_{k=1}^{N} \Omega_r\left(\frac{1}{n_k}\right),$$

where

$$\Omega_r(h) \leqslant \text{const.} \cdot \left\{ \omega_r(h) + \delta_r(f, h) \right\}. \tag{6}$$

For differentiable functions it follows:

Corollary 1. *Let $f \in AW^q H_r^{\omega_r}(\overline{D})$, $q \in \mathbb{N}_0$, $r \geqslant 2$, and ω_r be a normal majorant with exponent r satisfying the Stechkin condition (5). Let $f^{(r-1+q)}$ be continuous in a* *neighborhood of the vertices a_k, $k = 1, \dots, N$, and*

$$\sum_{k=1}^{N} d_k f^{(s)}(a_k) = 0 \quad \text{for } 0 \leqslant s \leqslant r - 1 + q.$$

Then for approximation with the quasipolynomial $\mathcal{P}_{q,n,r}(f)$ weighted with the generalized Jackson kernel

$$\| f - \mathcal{P}_{q,n,r}(f) \|_{AC(\overline{D})} \leqslant \text{const.} \sum_{k=1}^{N} \frac{1}{(n_k)^q} \cdot \Omega_r \left(\frac{1}{n_k} \right),$$

where Ω_r—a normal majorant with exponent r—satisfies inequality (6).

In Mel'nik's case $r = 1$ we get a stronger result, see Remark 1; namely, for $q \neq 0$ our proof allows to delete the term

$$\int_0^h \frac{\omega(f^{(q)}, t)}{t} \, dt$$

in the Zygmund condition in Theorem 1.

5. Proofs of Theorem 2 and Corollary 1

First we have closer look on the Leont'ev coefficients and their relation to Fourier coefficients. We use these properties for the subsequent proofs.

5.1. Leont'ev and Fourier coefficients

In [7], Mel'nik gave an important result establishing the relation between Fourier and Leont'ev coefficients for functions $f \in AC(\overline{D})$ and their first moduli of continuity. In [4] we extend his result to moduli of arbitrary order. We use this relation to reduce the Dirichlet series (1) to well-known Fourier series. Subsequently we can apply direct approximation theorems for Fourier series and deduce our new results.

Theorem 3 (*Forster [4]*). *Let ω_r be some normal majorant with exponent r and let $\frac{\omega_r(t)}{t}$ be integrable on $[0, \delta]$, $0 < \delta < 2\pi$. Let $f \in AH_r^{\omega_r}(\overline{D})$ with $\delta_r(f, h) < \infty$ and*

$$\sum_{k=1}^{N} d_k f^{(s)}(a_k) = 0 \quad \text{for all } 0 \leqslant s < r.$$

Then the Leont'ev coefficients $\kappa_f(\lambda_n^{(j)})$, $n \geqslant n(j)$, $j = 1, \dots, N$, are the Fourier coefficients of some 2π-periodic function of class $H_r^{\Omega_r}([0, 2\pi[)$ where

$$\Omega_r(h) \leqslant \text{const.} \left\{ \int_0^h \frac{\omega_r(u)}{u} \, du + h^r \int_h^{2\pi} \frac{\omega_r(u)}{u^{r+1}} \, du + \delta_r(f, h) \right\}.$$

For the proof of Corollary 1 in Section 5.3 we need a supplementary lemma:

Lemma 1 (*Mel'nik [8]*). *Let $f \in AC^q(\overline{D})$, $q \in \mathbb{N}$, and*

$$\sum_{k=1}^{N} d_k f^{(s)}(a_k) = 0 \quad \textit{for } s = 0, \dots, q-1.$$

Then the coefficients of the Dirichlet series of f have the form

$$\kappa_f(\lambda_m) = \frac{\kappa_{f^{(q)}}(\lambda_m)}{(\lambda_m)^q}.$$

This can be shown using integration by parts.

We now have all means to prove the results given in Section 4.

5.2. Proof of Theorem 2

We decompose the Dirichlet series of f with respect to property 3 of the zeros of L mentioned in Section 2:

$$\begin{aligned} f(z) = &\Bigg\{ \sum_{m=1}^{n_0} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)} \\ &+ \sum_{j=1}^{N} \sum_{m=n(j)}^{\infty} \kappa_f(\lambda_m^{(j)}) \left(\frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_{j+1}+a_j}{2}\right)} \right) \Bigg\} \\ &+ \sum_{j=1}^{N} B_j \sum_{m=n(j)}^{\infty} \kappa_f(\lambda_m^{(j)}) (-1)^m e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_{j+1}-a_j}{2}\right)} \\ =:& \,\Phi(z) + \sum_{j=1}^{N} \Phi_j(z). \end{aligned}$$

Due to the absolute convergence of the Dirichlet series and estimate 3 we also have absolute convergence of $\Phi(z)$ and $\Phi_j(z)$ for all $z \in D$.

The same decomposition is used for the quasipolynomial $\mathcal{P}_{0,n,r}(f)$:

$$\begin{aligned} &\mathcal{P}_{0,n,r}(f)(z) \\ &= \Bigg\{ \sum_{m=1}^{n_0} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)} + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) \kappa_f(\lambda_m^{(j)}) \\ &\quad \times \left(\frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_{j+1}+a_j}{2}\right)} \right) \Bigg\} \\ &\quad + \sum_{j=1}^{N} B_j \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) \kappa_f(\lambda_m^{(j)}) (-1)^m e^{\widetilde{\lambda}_m^{(j)}\left(z - \frac{a_{j+1}-a_j}{2}\right)} \\ &=: p_n(z) + \sum_{j=1}^{N} p_{j,n_j}(z). \end{aligned}$$

We define

$$F_j(w) = \sum_{m=n(j)}^{\infty} \kappa_f(\lambda_m^{(j)}) w^m \tag{7}$$

and

$$\Pi_{j,n_j}(w) = \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) \kappa_f(\lambda_m^{(j)}) w^m.$$

With property 1 of Section 2 we write

$$\begin{aligned}
\Phi_j(z) &= B_j \sum_{m=n(j)}^{\infty} \kappa_f(\lambda_m^{(j)}) (-1)^m e^{\left(\frac{2\pi m i}{a_{j+1}-a_j} + q_j e^{i\beta_j}\right)\left(z - \frac{a_{j+1}+a_j}{2}\right)} \\
&= B_j e^{q_j e^{i\beta_j}\left(z - \frac{a_{j+1}+a_j}{2}\right)} \sum_{m=n(j)}^{\infty} \kappa_f(\lambda_m^{(j)}) e^{\pi m i} e^{\frac{2\pi m i}{a_{j+1}-a_j}\left(z - \frac{a_{j+1}+a_j}{2}\right)} \\
&= B_j e^{q_j e^{i\beta_j}(z-a_j)} e^{-q_j e^{i\beta_j}\frac{a_{j+1}-a_j}{2}} \sum_{m=n(j)}^{\infty} \kappa_f(\lambda_m^{(j)}) e^{\frac{2\pi m i}{a_{j+1}-a_j}(z-a_j)} \\
&= B_j e^{q_j e^{i\beta_j}(z-a_j)} e^{-q_j e^{i\beta_j}\frac{a_{j+1}-a_j}{2}} F_j\left(e^{\frac{2\pi i}{a_{j+1}-a_j}(z-a_j)}\right)
\end{aligned}$$

and

$$p_{j,n_j}(z) \quad = \quad B_j e^{q_j e^{i\beta_j}(z-a_j)} e^{-q_j e^{i\beta_j}\frac{a_{j+1}-a_j}{2}} \Pi_{j,n_j}\left(e^{\frac{2\pi i}{a_{j+1}-a_j}(z-a_j)}\right). \tag{8}$$

With (5), (7) and Theorem 3 we deduce $F_j(e^{i\theta}) \in H_r^{\Omega_r}([0, 2\pi])$ (compare with [1]). Hence for the approximation with the generalized Jackson kernel we obtain by Stechkin's theorem [11, Theorem 11], [2, Chapter 7, Theorem 2.3]

$$|F_j(w) - \Pi_{j,n_j}(w)| \leqslant \text{const.} \cdot \Omega_r\left(\frac{1}{n_j}\right) \quad \text{for } |w| = 1.$$

$F_j(w) - \Pi_{j,n_j}(w)$ is holomorphic on $\{w : |w| < 1\}$. Thus this function reaches its maximum on the boundary of this domain. Hence

$$|F_j(w) - \Pi_{j,n_j}(w)| \leqslant \text{const.} \cdot \Omega_r\left(\frac{1}{n_j}\right) \quad \text{for } |w| \leqslant 1.$$

The estimates $|e^{2\pi i \frac{z-a_j}{a_{j+1}-a_j}}| \leqslant 1$ for all $z \in \overline{D}$ and $|e^{q_j e^{i\beta_j}(z-a_j)}| \leqslant \text{const.}$ for all $z \in \overline{D}$ lead to

$$|\Phi_j(z) - p_{j,n_j}(z)| \leqslant \text{const.} \cdot \Omega_r\left(\frac{1}{n_j}\right) \quad \text{for all } z \in \overline{D}.$$

Now we consider the remaining term $\Phi(z) - p_n(z)$. With property 3 in Section 2 we infer

$$\begin{aligned}
&|\Phi(z) - p_n(z)| \\
&\quad = \Bigg| \sum_{j=1}^{N} \sum_{m=n_j+1}^{\infty} \kappa_f(\lambda_m^{(j)}) \left(\frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_{j+1}+a_j}{2}\right)} \right) \\
&\qquad + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} x_{n_j,r,m}\, \kappa_f(\lambda_m^{(j)}) \left(\frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_{j+1}+a_j}{2}\right)} \right) \Bigg| \\
&\quad \leqslant \sum_{j=1}^{N} \sum_{m=n_j+1}^{\infty} |\kappa_f(\lambda_m^{(j)})|\, A(0) e^{-c_2 m} \\
&\qquad + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} |x_{n_j,r,m}|\, |\kappa_f(\lambda_m^{(j)})|\, A(0) e^{-c_2 m}.
\end{aligned}$$

It is

$$x_{n_j,r,m} = \mathcal{O}\left(\frac{m^r}{n_j^r}\right) \tag{9}$$

for $n_j \to \infty$, and thus for all $z \in \overline{D}$ and some appropriate constant $c > 0$

$$\begin{aligned}
|\Phi(z) - p_n(z)| &\leqslant \text{const.} \sum_{j=1}^{N} \left(e^{-cn_j} + \frac{1}{n_j^r} \right) \\
&\leqslant \text{const.} \sum_{j=1}^{N} \Omega_r\left(\frac{1}{n_j}\right).
\end{aligned}$$

This proves Theorem 2.

5.3. *Proof of Corollary 1*

We carry out an analogous decomposition as in the proof of Theorem 2. From Theorem 3, Lemma 1 and (5) [1] we infer $F_j(e^{i\theta}) \in W^q H_r^{\Omega_r}([0, 2\pi])$. Hence for the approximation with the generalized Jackson kernel

$$|F_j(w) - \Pi_{j,n_j}(w)| \leqslant \text{const.} \frac{1}{n_j^q}\, \Omega_r\left(\frac{1}{n_j}\right) \quad \text{for } |w| \leqslant 1,$$

because $F_j(w) - \Pi_{j,n_j}(w)$ attains the maximum on the boundary of the domain $\{w : |w| \leqslant 1\}$. Thus

$$|\Phi_j(z) - p_{j,n_j}(z)| \leqslant \text{const.} \frac{1}{n_j^q}\, \Omega_r\left(\frac{1}{n_j}\right) \quad \text{for all } z \in \overline{D}.$$

With property 3 of Section 2 and Eq. (9) we infer

$$
\begin{aligned}
&|\Phi(z) - p_n(z)| \\
&\quad = \left| \sum_{j=1}^{N} \sum_{m=n_j+1}^{\infty} \kappa_f(\lambda_m^{(j)}) \left(\frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_{j+1}+a_j}{2}\right)} \right) \right. \\
&\qquad \left. + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} x_{n_j,r,m}^{q+1} \, \kappa_f(\lambda_m^{(j)}) \left(\frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} - (-1)^m B_j e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_{j+1}+a_j}{2}\right)} \right) \right| \\
&\quad \leqslant \sum_{j=1}^{N} \sum_{m=n_j+1}^{\infty} |\kappa_f(\lambda_m^{(j)})| \, A(0) e^{-c_2 m} \\
&\qquad + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} |x_{n_j,r,m}|^{q+1} \, |\kappa_f(\lambda_m^{(j)})| \, A(0) e^{-c_2 m} \\
&\quad \leqslant \text{const.} \sum_{j=1}^{N} \left(e^{-cn_j} + \frac{1}{n_j^{r(q+1)}} \right) \\
&\quad \leqslant \text{const.} \sum_{j=1}^{N} \Omega_r \left(\frac{1}{n_j} \right)
\end{aligned}
$$

for some constant $c > 0$, and the claim is proved.

Remark 1. The proof of Theorem 1 can be deduced as a special case of the proofs above.

Moreover, let $f \in AC^{(r-1)}(\overline{D})$, $r > 1$. Let $\eta : [0, 1] \to [a_k, a_j]$, $\eta(t) = a_j - (1 - t)(a_j - a_k)$, be a continuous parametrization of the straight-line interval $[a_k, a_j]$. Then for $k \neq j + 1$

$$
\begin{aligned}
&(f - P_{j,k}) \circ \eta(u) \\
&\quad = \int_0^u \int_0^{u_1} \ldots \int_0^{u_{r-2}} (f \circ \eta)^{(r-1)}(v) - (f \circ \eta)^{(r-1)}(0) \, dv \, du_{r-2} \ldots du_1 \\
&\qquad - u^r \int_0^1 \int_0^{u_1} \ldots \int_0^{u_{r-2}} (f \circ \eta)^{(r-1)}(v) \\
&\qquad - (f \circ \eta)^{(r-1)}(0) \, dv \, du_{r-2} \ldots du_1
\end{aligned} \tag{10}
$$

and for $k = j + 1$

$$
\begin{aligned}
&(f - P_{j,j+1}) \circ \eta(u) \\
&\quad = \int_0^u \int_0^{u_1} \ldots \int_0^{u_{r-2}} (f \circ \eta)^{(r-1)}(v) - (f \circ \eta)^{(r-1)}(0) \, dv \, du_{r-2} \ldots du_1 \\
&\qquad - u^r \int_0^1 \int_0^{u_1} \ldots \int_0^{u_{r-2}} (f \circ \eta)^{(r-1)}(v) - (f \circ \eta)^{(r-1)}(0) \, dv \, du_{r-2} \ldots du_1 \\
&\qquad - u^r (u - 1) Q_{r-1}(u)
\end{aligned} \tag{11}
$$

for some polynomial Q_{r-1} of degree $r-1$. Thus in both cases

$$|(f-P_{j,k})\circ\eta(u)|\leqslant \text{const.}\, u^{r-1}\omega_1(f^{(r-1)},u),$$

and

$$\begin{aligned}\delta_r(f,h) &\leqslant \text{ const. } \left\{\int_0^h \frac{u^{r-1}\omega_1(f^{(r-1)},u)}{u}\,du + h^r\int_h^{2\pi}\frac{u^{r-1}\omega_1(f^{(r-1)},u)}{u^{r+1}}\,du\right\}\\ &\leqslant \text{ const. } h^r\int_h^{2\pi}\frac{\omega_1(f^{(r-1)},u)}{u^2}\,du.\end{aligned}$$

Thus Theorem 1 follows from Theorem 2 and by the inequality $\omega_r(f,u)\leqslant$ const. u^{r-1} $\omega_1(f^{(r-1)},u)$.

6. Example

Consider the function

$$g(z)=(z+1)\ln\left(\frac{1}{z+1}\right)$$

on the square $D\subset\mathbb{C}$ with vertices $-1+i$, $-1-i$, $1-i$ and $1+i$. Then

$$g'(z)=\ln\left(\frac{1}{z+1}\right)-1.$$

The derivative has a logarithmic branch point at $z=-1$ but is continuous at the four vertices (see the absolute values of both functions in Fig. 1).

By direct calculation we see $\omega_{1,\overline{D}}(g,h)_\infty=\mathcal{O}(h\ln\frac{1}{h})$ for $h\to 0$. Thus $\omega_{1,\overline{D}}(g,h)$ does not satisfy Zygmund's condition and Mel'nik's Theorem 1 cannot be applied.

Considering the second moduli of smoothness we get $\omega_{2,\overline{D}}(g,h)_\infty=\mathcal{O}(h)$ and $\delta_2(g,h)=\mathcal{O}(h)$ for $h\to 0$, and Stechkin's condition is complied. Applying our new Theorem 2 we obtain

$$\|g-\mathcal{P}_{0,n,2}(g)\|_{AC(\overline{D})}=\mathcal{O}\left(\omega_{2,\overline{D}}\left(g,\frac{1}{n_*}\right)+\delta_2\left(g,\frac{1}{n_*}\right)\right)=\mathcal{O}\left(\frac{1}{n_*}\right)$$

for $n_*\to\infty$. Here $n_*=\min\{n_1,\ldots,n_N\}$ denotes the minimal component of the multi-index $n=(n_1,\ldots,n_N)\in\mathbb{N}^N$. This shows that Theorem 2 sharpens the results on the order of approximation with quasipolynomials of Jackson's type.

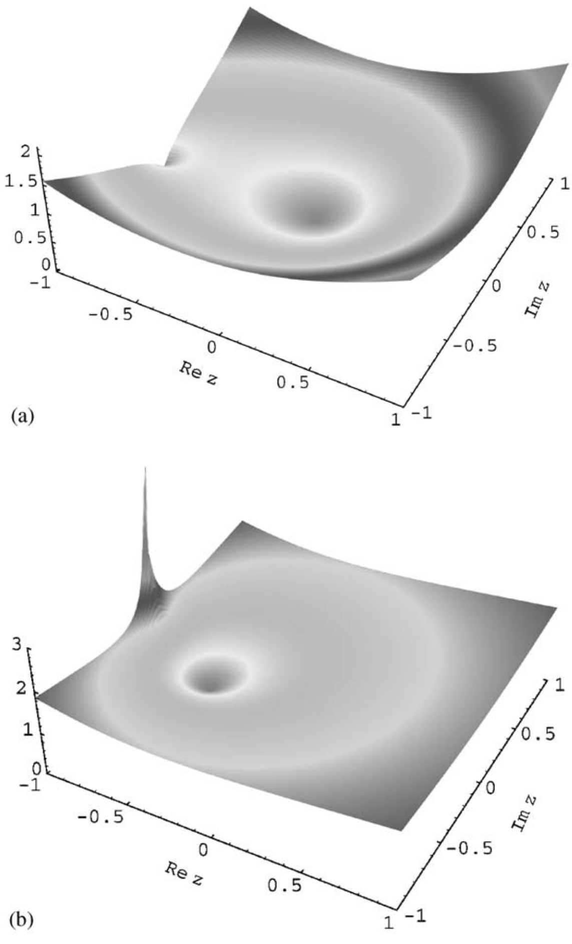

Fig. 1. (a) Absolute value of the function $g(z) = (z+1)\ln\left(\frac{1}{z+1}\right)$ on the square D. (b) Absolute value of its derivative g' on D with a logarithmic branch point at $z = -1$.

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